Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

1. Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. *For example, create a story context for (2/3) ÷ (3/4) and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that (2/3) ÷ (3/4) = 8/9 because 3/4 of 8/9 is 2/3. (In general, (a/b) ÷ (c/d) = ad/bc.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?*

Connections to Prior Grades

CCSM 5th Grade Content introduces division of whole number by unit fraction OR a unit fraction by a whole number. 6th Grade students extend understandings of multiplication and division to include ***fraction ÷ fraction*** using a) visual fraction models, b) equations, c) computation, and d) interpretations.

The Relationship Between Multiplication and Division

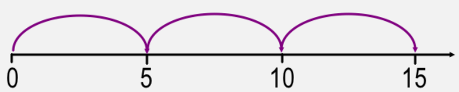
Table 2 (page 89) in the CCSSM document provides examples of common multiplication and division situations and interpretations.

Multiplication: *Repeated addition of same sized groups*

The Commutative Property allows factors to be arranged in any order without changing the meaning of the product, though the pictorial representation will look differently.

3 × 5 = 15

number of groups	×	items in each group	=	total number of items
3	×	5	=	15



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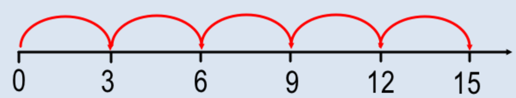


Division: *Finding the missing factor, or repeated subtraction*

Division 'undoes' multiplication. Division begins with the total number of items. The items are either divided into the 'number of groups' or the 'items in each group'. Without context, students simply compute a quotient. With context, students determine if they are finding the 'number of groups' or the 'items in each group'. Thus, they are ***interpreting the quotient*** as partitive (finding the 'items in each group') or measurement (finding the 'number of groups'). Measurement can also be called quotitive. It is the variety of problem structures, not the mastery of 'partitive' and 'measurement' vocabulary, that builds understanding of division and how visual models are drawn when solving division problems.

Sample Problem

15 ÷ 5 = 3

Context determines the meaning, or interpretation, of the 5 and 3.

Partitive Interpretation

Jack ran the same distance for 5 days. He ran a total of 15 miles. How many miles did he run each day?

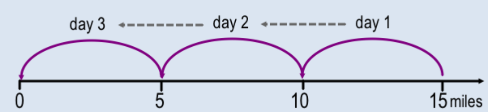
Known: total items (*15 miles*) and number of groups (*5 days*)
Unknown: number of items in each group (*how many miles each day*)
Equation: 15 ÷ 5 = 3
Visual Fraction Model (number line model)

Solution: 15 miles divided into 5 groups is 3 miles in each group. *Jack ran 3 miles each day.*

Measurement (or Quotitive) Interpretation

Jane likes to run 5 miles per day. Last week, she ran 15 miles. How many days did Jane run last week?

Known: total items (*15 miles*) and size of group (*5 miles*)
Unknown: number of groups (*how many days*)
Equation: 15 ÷ 5 = 3
Visual Fraction Model (number line model)



Solution: 15 miles divided into 5 miles per group is 3 groups. *Jane ran 5 days last week.*

Division of Fractions by Fractions: computing, modeling, interpreting

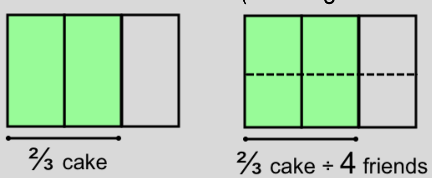
6th Grade students will use visual fraction models (number line/bar model, set model, area/regional model), equations, and their understanding of multiplication and division to compute and interpret quotients. Below is a recommended teaching sequence that scaffolds the mathematics for students. For each sequence stage, a partitive and a measurement example is shown using a variety of visual fraction models. Number lines, bar models, and area models are generally easier to use for visual models when dividing fractions, and students need multiple opportunities to *draw* these diagrams.

STAGE 1 Fraction ÷ Whole Number

Partitive Interpretation

Four friends share ⅔ of a cake. How much of the cake does each friend get?

Known: total items *(⅔ cake)* and number of groups (*4 friends*)
Unknown: number of items in each group (*how much each friend gets*)
Equation: ⅔ ÷ 4 = ?
Visual Fraction Model (area/regional model)



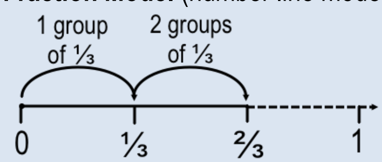
It is important to remember that the 'total items' was less than 1, but the whole cake is the *referent whole*. While each friend will get one of the four *pieces* of ⅔, the *size* of the piece is ⅙ of the cake.

Solution: *Each friend gets ⅙ of the cake.*

Measurement (or Quotitive) Interpretation

Karen has ⅔ meters of ribbon. One bow takes ⅓ meters of ribbon. How many bows can be made?

Known: total items *(⅔ meters*) and size of groups (⅓ meters of ribbon)
Unknown: number of groups (*how many bows can be made*) OR 'How many ⅓ in ⅔?
Equation: ⅔ ÷⅓ = ?
Visual Fraction Model (number line model)



There are 2 groups of ⅓ in ⅔.
2 bows can be made.

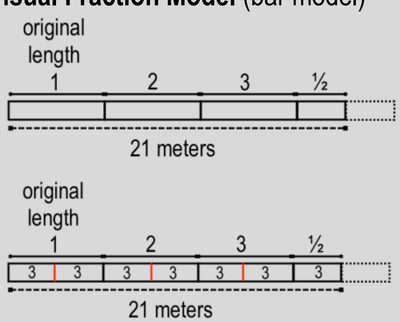
Solution: *Karen can make 2 bows.*

STAGE 2 Whole Number ÷ Fraction

Partitive Interpretation

A bungee cord was stretched 3½ times its original length. When stretched it was 21 meters long. What is the original length of the bungee cord?

Known: total items *(21 meters*) and number of groups (*3½*)
Unknown: number of meters in each group
Equation: 21 ÷ *3½* = ?
Visual Fraction Model (bar model)



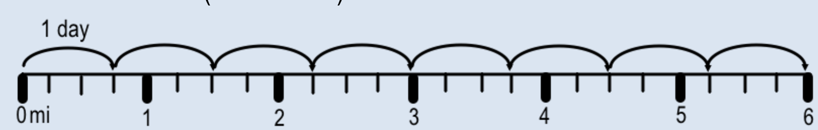
One solution strategy illustrates why mixed numbers are converted to fractions before computation. By making the group sizes equal (in this context making all groups halves) it is easier to 'fair share' the 21 meters. Once the 'whole' groups have been made halves, there are 7 groups. 21 ÷ 7 = 3 There are 3 meters in each 'half' group, so there are 6 meters in one 'whole' group.

Solution: *The original length of the bungee cord is 6 meters.*

Measurement (or Quotitive) Interpretation

Jonah wants to run 6 miles. He runs ¾ miles each day. How many days will it take him to run the 6 miles?

Known: total items (*6 miles*) and items per group (*¾ miles each day*)
Unknown: number of groups (*days*)
Equation: 6 ÷ ¾ = ?
Visual Fraction Model (number line)



There are 8 groups of ¾ in 6.

Solution: *It will take Jonah 8 days to run 6 miles if he runs ¾ miles each day.*

STAGE 3 Fraction ÷ Fraction

At this stage, it is recommended to scaffold learning through careful selection of fractions. Many students develop stronger understanding as they progress from 1) fractions that have the same denominator (benchmark denominators are simplest – halves, thirds, fourths, sixths, eighths, etc…) **AND** fractions where one denominator is a multiple of the other, to 2) fractions with different denominators that may or may not be multiples of the other. The examples below follow this sequence. It is still important to provide students with a balance of partitive and measurement division problems.

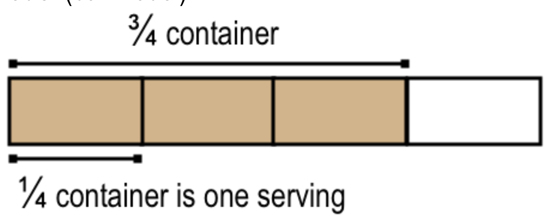
Example 1 *I have a container of yogurt that is ¾ full. One serving of yogurt is ¼ of the container. How many servings are in the container?*

Known: total items (*¾ yogurt container*) and size of 1 serving (*¼ of the container*)

Unknown: number of groups (*servings)* in the container: "How many servings are in the container?" OR "How many ¼'s are in ¾?"

Equation: ¾ ÷ ¼ = ?

Visual Fraction Model (bar model)



Solution: *There are 3 servings in the container.*

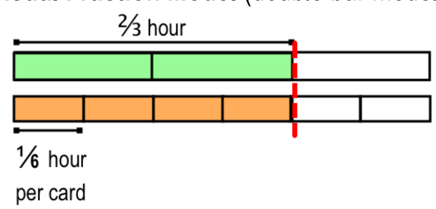
Example 2 *Max worked for ⅔ hour. It takes ⅙ hour to make one card. How many cards has Max made?*

Known: total time (*⅔ hour*) and amount of time per card (*⅙ hour*)

Unknown: number of groups (*cards*) that can be made in the remaining time: "How many cards can be made?" OR "How many ⅙ are in ⅔?"

Equation: ⅔ ÷ ⅙ = ?

Visual Fraction Model (double bar model)



When doing a double bar model, it is important to represent the same referent whole. In this case, the whole is one hour for both the time worked and the time it takes to make one card. There are 4 groups of ⅙ in ⅔.

Solution: *Max has made 4 cards.*

Example 3 (this is one of the example problems given in the CCSSM document)

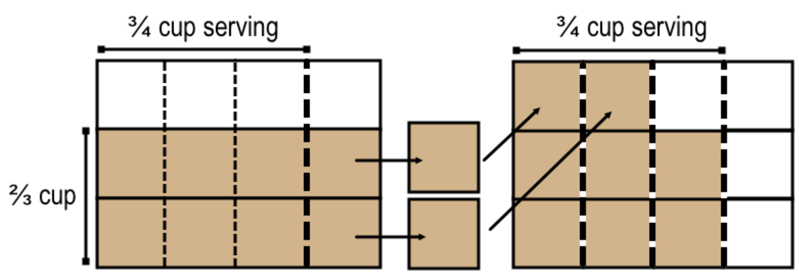
How many ¾ cup servings are in ⅔ cup of ice cream?

Known: total items (*⅔ cup ice cream*) and size of 1 serving (*¾ cup ice cream*)

Unknown: number of groups (*servings)* in the ⅔ cup: "How many servings are in the container?" OR "How many ¾'s are in ⅔?"

Equation: ⅔ ÷ ¾ = ?

Visual Fraction Model (area model)



In this problem, the referent whole is *one serving*, **not** one container. One serving is ¾ cup. There are 8/9 servings.

Solution: *There are 8/9 servings in the container.*

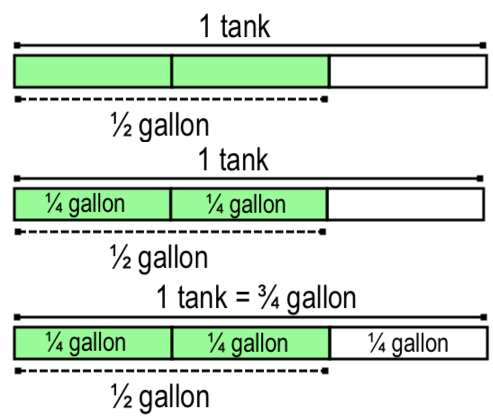
Example 4 *A lawn mower's tank is ⅔ full. It has ½ gallon in it. How many gallons does it hold?*

Known: total items (*½ gallon*) and number of groups (⅔ tank)

Unknown: size of group (how many gallons the tank holds)

Equation: ½ ÷ ⅔ = ?

Visual Fraction Model (bar model)



The first bar model shows what is known in the problem.

The second bar model shows ½ ÷ 2, which is the size of each group.

Because the tank is the referent whole AND it is measured in thirds, the last third is also ¼ gallon.

Solution: *The tank holds ¾ of a gallon.*

Connecting 'Invert and Multiply' to Relationships of Multiplication and Division

One method to illustrate the reason why 'invert and multiply' works is to 1) give a measurement and partitive context to a problem, 2) solve each using a sequence of pictures, then 3) articulate and discuss where and when the multiplication and division take place. In the example below, the eight is either multiplied first, then divided into groups (measurement) OR the eight is divided first, then multiplied. It is helpful to scaffold reasoning about the algorithm by using whole number ÷ fraction first.

Example $8 \div \frac{2}{3} = 12$

MEASUREMENT (QUOTITIVE) EXAMPLE

Jill has 8 cups of flour. One batch of cookies requires ⅔ cup. How many batches of cookies can be made?

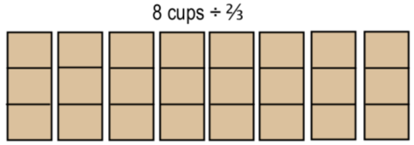
One way to approach this problem is to ask 'How many groups of ⅔ cup are in 8 cups?'. However, when asked to explain why invert and multiply works in this context, a sequence of drawings helps illustrate what is happening to the eight cups as it is multiplied by 3 then divided by 2.

The first stage of the drawing shows 8 cups.

8 cups

The next stage of the drawing shows the 8 cups being put into thirds, making eight 'groups of thirds' (8 × 3). At the end, there is a total of 24 one-thirds.

8 cups ÷ ⅔



The last stage of the drawing shows the 24 thirds being put into groups of 2, or the *size* of each group; each group, or batch, is ⅔ cup.

8 cups ÷ ⅔ cup per batch = 12 batches

8 x 3 = total items (24 thirds)

$$8 \times \frac{3}{2} = \frac{8 \times 3}{2} = \frac{24}{2} = 12$$

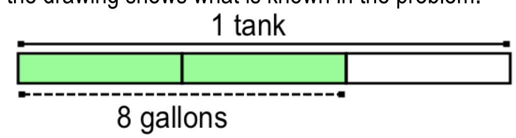
size of groups

PARTITIVE EXAMPLE

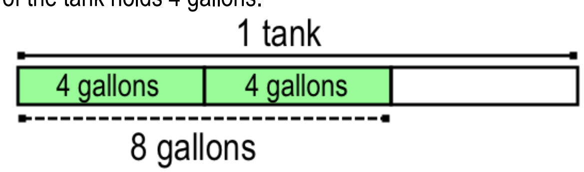
The car's gas tank has 8 gallons in it. The tank is ⅔ full. How many gallons does the tank hold?

The question 'How many groups of ⅔ are in 8?' does not work for this context. This sequence of drawings helps illustrate what is happening to the 8 gallons as they are divided by 2 then multiplied by three.

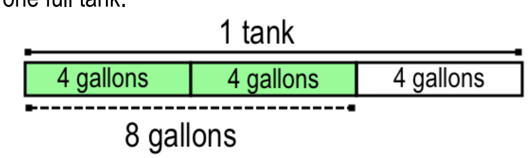
The first stage of the drawing shows what is known in the problem.



The next stage of the drawing shows the 8 gallons being divided by two to determine that each third of the tank holds 4 gallons.



The last stage of the drawing shows the 4 being multiplied by three to fill each third of the tank, making one full tank.



8 ÷ 2 = size of group (4 gallons)

number of groups (3)

$$8 \times \frac{3}{2} = \frac{8 \times 3}{2} = 4 \times 3 = 12 \text{ gallons}$$